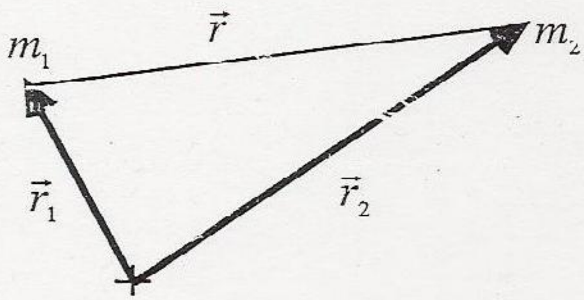
The 2-body Problem

The general problem of the orbits of 2 bodies with similar masses, e.g. the Earth-Moon system or a double-star is both of direct and indirect importance to us. Indirectly, because it can be reduced to a 1-body problem for the relative motion, and so, we can economically consider both at the same time. Begin by reviewing the standard decoupling in center-of-mass and relative motions.

Equations of motion:



$$m_1\ddot{\vec{r}}_1 = -Gm_1m_2\,\frac{\vec{r}_1-\vec{r}_2}{r^3},$$

$$m_2\ddot{\vec{r}}_2 = -Gm_1m_2\,\frac{\vec{r}_2-\vec{r}_1}{r^3},$$

Add these equations: $m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = 0$

Integrate: $m_1\vec{v}_1 + m_2\vec{v}_2 = \vec{a} = \text{constant vector}$

Integrate again: $m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{a}t + \vec{b}$

Let $M = m_1 + m_2$ *and* $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$, *then* $M\vec{R} = \vec{a}t + \vec{b}$.

I.e. Uniform Center-of-Mass Motion.

Relative Motion:

Consider a new (moving) coordinate system with its origin at the CM.

$$\vec{R} = 0, \quad \vec{r}_1 = \frac{-m_2}{m_1}\vec{r}_2.$$

$$\text{Then,} \quad \ddot{\vec{r}}_1 = \frac{-Gm_2}{r^3}\left(\vec{r}_1 + \frac{m_1}{m_2}\vec{r}_1\right) = -GM\frac{\vec{r}_1}{r^3},$$

$$\ddot{\vec{r}}_2 = -GM\frac{\vec{r}_2}{r^3}.$$

Subtract to get – $\ddot{\vec{r}} = \dfrac{-GM}{r^3}\vec{r}$.

Now consider just the radial part of the force eq. for the relative motion (gravity acts radially), in spherical coordinates.

$$a_r = \quad \ddot{r} - r\dot{\phi}^2 = \frac{-GM}{r^2}.$$

(Symon)

Also have,

$$\frac{L}{m} = \quad r^2\dot{\phi} = |\vec{h}| = h = \text{constant},$$

$$\text{so} \quad \dot{\phi}^2 = \frac{h^2}{r^4}.$$

Substituting this into the force eq.,

$$\ddot{r} - \frac{h^2}{r^3} = \frac{-GM}{r^2}.$$

Now multiply by $\dot{r}$ and integrate term by term.

If this is written in the form:

$$\dot{\vec{r}} = \vec{v},$$
$$\dot{\vec{v}} = \frac{-GM}{r^3}\vec{r},$$

then it is seen to be a system of 6 first-order ordinary differential equations.

Thus, we expect <u>6 constants of integration</u>. Three of these can be associated with the angular momentum vector.

Proof: $\vec{r} \; x \; \ddot{\vec{r}} = 0$

but, $\dfrac{d}{dt}(\vec{r} \; x \; \dot{\vec{r}}) = \dot{\vec{r}} \; x \; \dot{\vec{r}} + \vec{r} \; x \; \ddot{\vec{r}} = 0$

$\therefore \quad \vec{r} \; x \; \dot{\vec{r}} = \vec{h}$

= specific angular momentum

= constant

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{h^2}{r^2} - \underbrace{\frac{GM}{r}}_{\text{P.E.}} = \text{const.} = \mathcal{E}$$

First 2 terms $= \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 = \frac{1}{2}\dot{\vec{r}}\cdot\dot{\vec{r}} = $ K.E.

Conservation of energy!

Orbital Sol'n.

Consider classical turning points where $\dot{r}=0$.

$$r^2 + \frac{GM}{\mathcal{E}}r - \frac{1}{2}\frac{h^2}{\mathcal{E}} = 0$$

Quadratic has 2 sol'ns. r_+, r_-, inner and outer turning points.

$\therefore$ Equation above is equivalent to $(r-r_+)(r-r_-)=0$, and energy eq. can be written

$$-\frac{r^2\dot{r}^2}{2\mathcal{E}} + (r-r_+)(r-r_-) = 0$$

or $$\dot{r}^2 = \frac{2\mathcal{E}(r_+-r)(r-r_-)}{r^2}$$

Nasty looking nonlinear ODE.

Don't do in class.

x

To get solution in polar coord.s change variable from t to ϕ:

$$\dot{r} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{h}{r^2}\frac{dr}{d\phi}$$

So the ODE becomes,

$$\frac{dr}{d\phi} = \sqrt{\frac{2\mathcal{E}}{h^2}}\; r\left[(r_+ - r)(r - r_-)\right]^{1/2}$$

Integrate:

$$\int \frac{dr}{r\left[(r_+ - r)(r - r_-)\right]^{1/2}} = \sqrt{\frac{2\mathcal{E}}{h^2}} \int d\phi$$

Both integrals can be evaluated explicitly, for LHS see CRC 258.

$$LHS = \left\{\frac{1}{\sqrt{r_+ r_-}} \sin^{-1}\left[\frac{(r_- + r_+)r - 2r_+ r_-}{|r|\sqrt{(r_- + r_+)^2 - 4r_+ r_-}}\right]\right\}_{r_-}^{r}$$

$$= \frac{1}{\sqrt{r_+ r_-}}\left\{\sin^{-1}\left[\frac{(r_- + r_+)r - 2r_+ r_-}{|r|\,(r_+ - r_-)}\right] - \underbrace{\sin^{-1}(-1)}_{= (-\pi/2)}\right.$$

$$RHS = \sqrt{\frac{2\mathcal{E}}{h^2}}\left(\phi - \underbrace{\phi(r_-)}_{= -\pi/2}\right)$$

Substitute these results and take the sin of both sides.

$$\frac{(r_- + r_+)r - 2r_+r_-}{|r|\,(r_+ - r_-)} = \sin\left[\sqrt{\frac{2\varepsilon r_+ r_-}{h^2}}\left(\phi + \tfrac{\pi}{2}\right) - \tfrac{\pi}{2}\right]$$

$$= -\cos\left[\sqrt{\frac{2\varepsilon r_+ r_-}{h^2}}\left(\phi + \tfrac{\pi}{2}\right)\right]$$

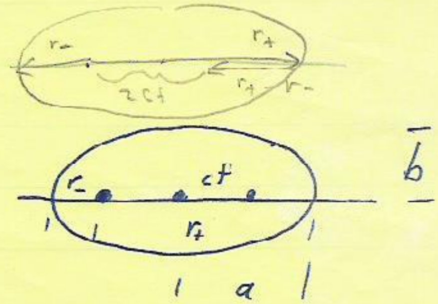
Solve for r (note: $|r| = r$):

$$r = \frac{2\,\frac{r_+ r_-}{(r_+ + r_-)}}{1 + \frac{(r_+ - r_-)}{(r_+ + r_-)}\cos[\cdot]},$$

which is essentially the same as the standard conic section formula.

$$r = \frac{a(1-e^2)}{\underbrace{1 + e\cos(\nu)}_{\text{azimuth angle}}}$$

with semi-major axis a, eccentricity e.



Note: $r_+ + r_- = 2a$

Center to focus distance $= \sqrt{a^2 - b^2}$

and $r_+ - r_- = 2$ c.-f. distance $= 2\sqrt{a^2 - b^2}$

Therefore, $\dfrac{r_+ - r_-}{r_+ + r_-} = \dfrac{\cancel{2}\sqrt{a^2-b^2}}{\cancel{2}a} = e$ (by definition)

The following important relations can also be derived,

$$\boxed{\mathcal{E} = \frac{-GM}{2a}} \qquad \boxed{h^2 = GMa(1-e^2)}$$

(Remember: $\mathcal{E}$ is the energy per unit mass of the relative orbit and h is the angular momentum per unit mass.)

In general, to specify a 2-body orbit we need 6 integration constants. Can use $\mathcal{E}$, $\vec{h}$, r_+, r_-, but these are theoretical quantities, which cannot be directly determined observationally. A more traditional set which relates more directly to observables consists of....

ω - argument of apsides or pericenter. Specifies orientation of orbit in its own plane.

a - semi-major axis

e - eccentricity

Ω - longitude of ascending node. Specifies the line of nodes in the fundamental plane.

i - inclination of the orbital plane relative to the fundamental plane.

T - time (or phase) of periapse passage. Specifies where the particle is located on its orbit.

Note: the definitions of the angles are somewhat different in binary star studies. (This because we observe such systems with an arbitrary projection angle on the sky.)

Note 2: as Gauss pointed out there are really 7 parameters, including the total mass (or period).

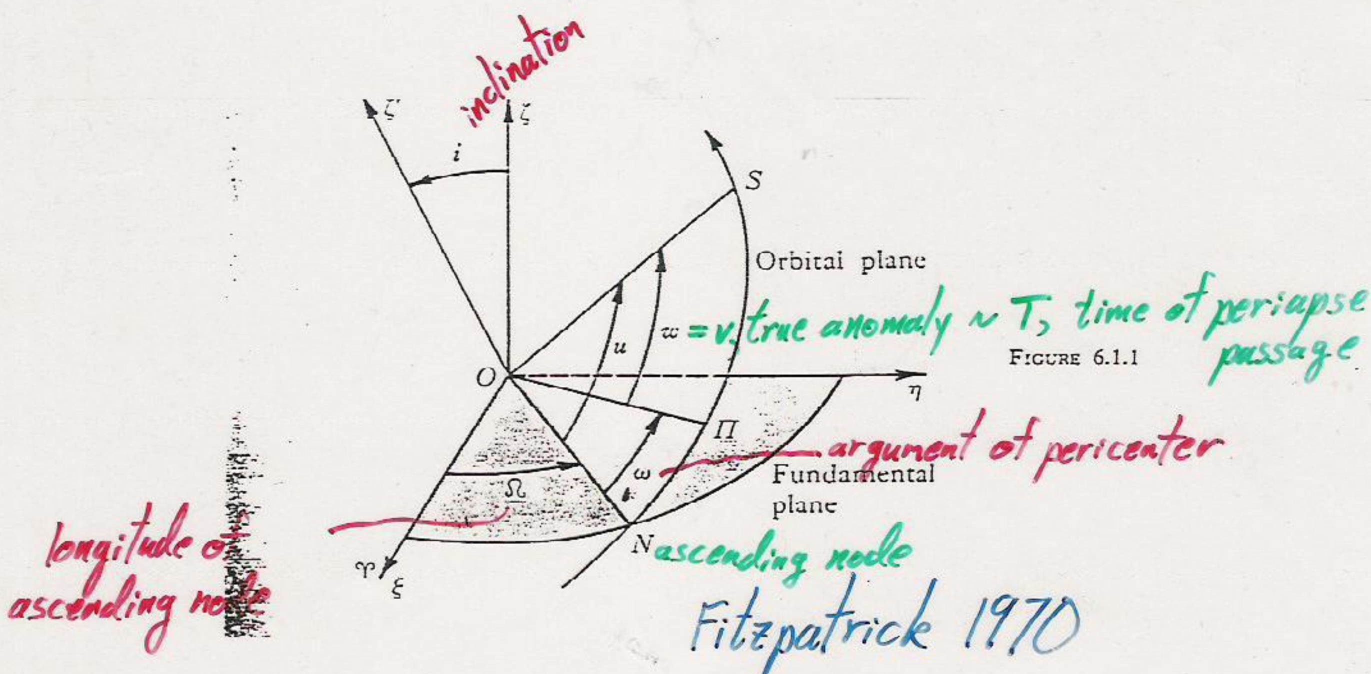


Figure 6.1.1

Fitzpatrick 1970

The Restricted 3-body ~~Program~~ Problem

The general 3-body problem is highly complex, has no complete solution, and is of much interest in celestial mechanics. See any C.M. text.

We will limit ourselves to the special case where one body is much less massive than the other two. Will consider two examples:

① The even more restricted case where the "massless" body is closely orbiting the most massive body, e.g. lunar perturbations to an artificial satellite orbit. Will do this example qualitatively, using pictures (from Taff). Also assume circular orbits initially, and a stationary moon.

It can be shown that the magnitude of the perturbations (in fractional radius $\frac{\Delta r}{r}$ or velocity $\frac{\Delta v}{v}$) is of order

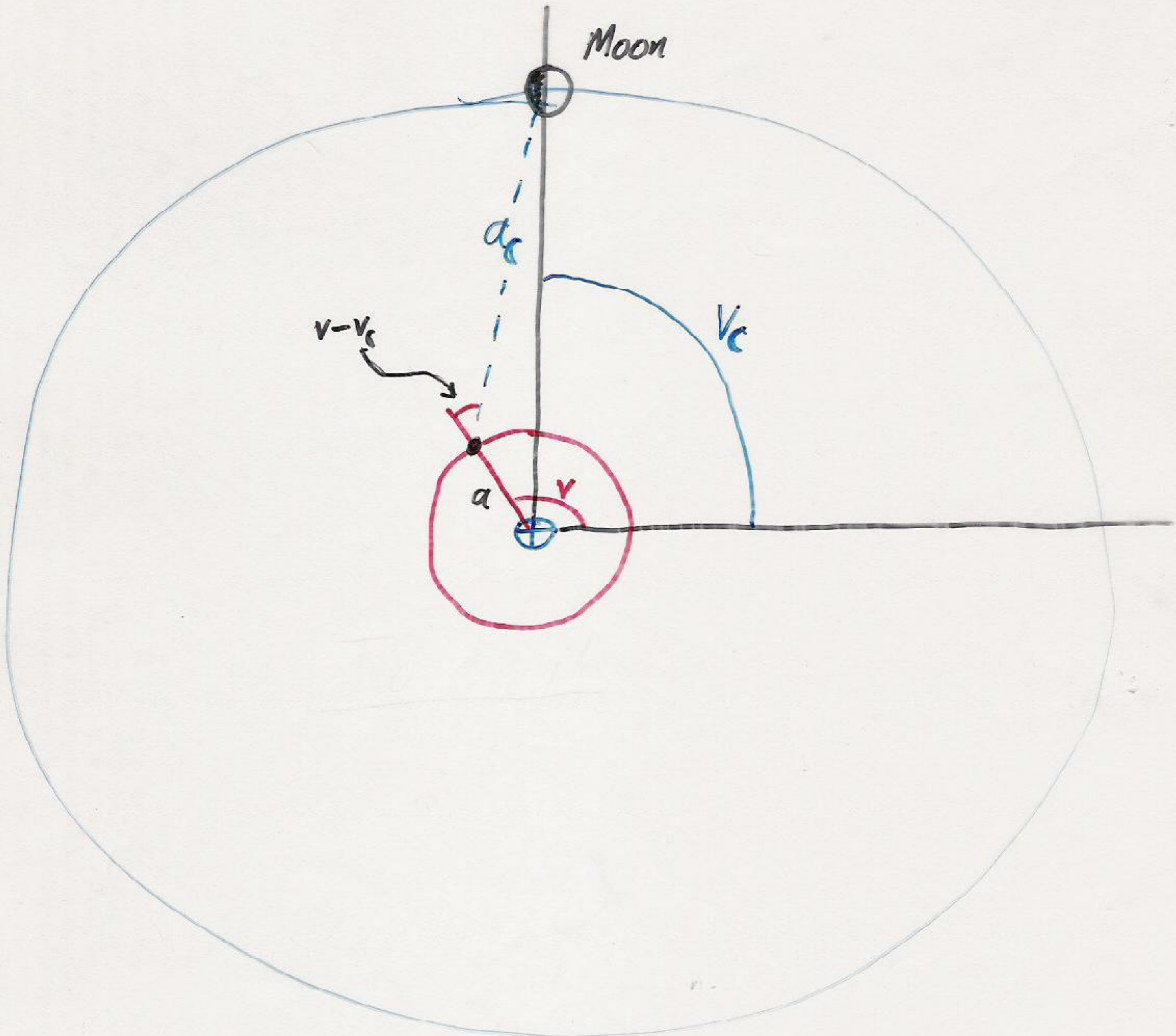
$$\varepsilon = \underbrace{\left(\frac{GM_{\text{g}}}{GM_{\oplus}}\right)}_{\text{little}}\underbrace{\left(\frac{a}{a_{\text{g}}}\right)^2}_{\text{little squared}},$$

i.e. $r \simeq a + \underbrace{\varepsilon r_1(t)}_{\text{time-varying part of order unity}}$

Examples:

	a	a/a_{g}	ε
L.E.O.	6600 km	0.017	3.7×10^{-6}
$2r_{\oplus}$	13,000 km	0.034	1.4×10^{-5}
	$\frac{1}{2} a_{\text{g}}$	$\frac{1}{2}$	0.0031

x



The Restricted 3-body Problem in the Plane

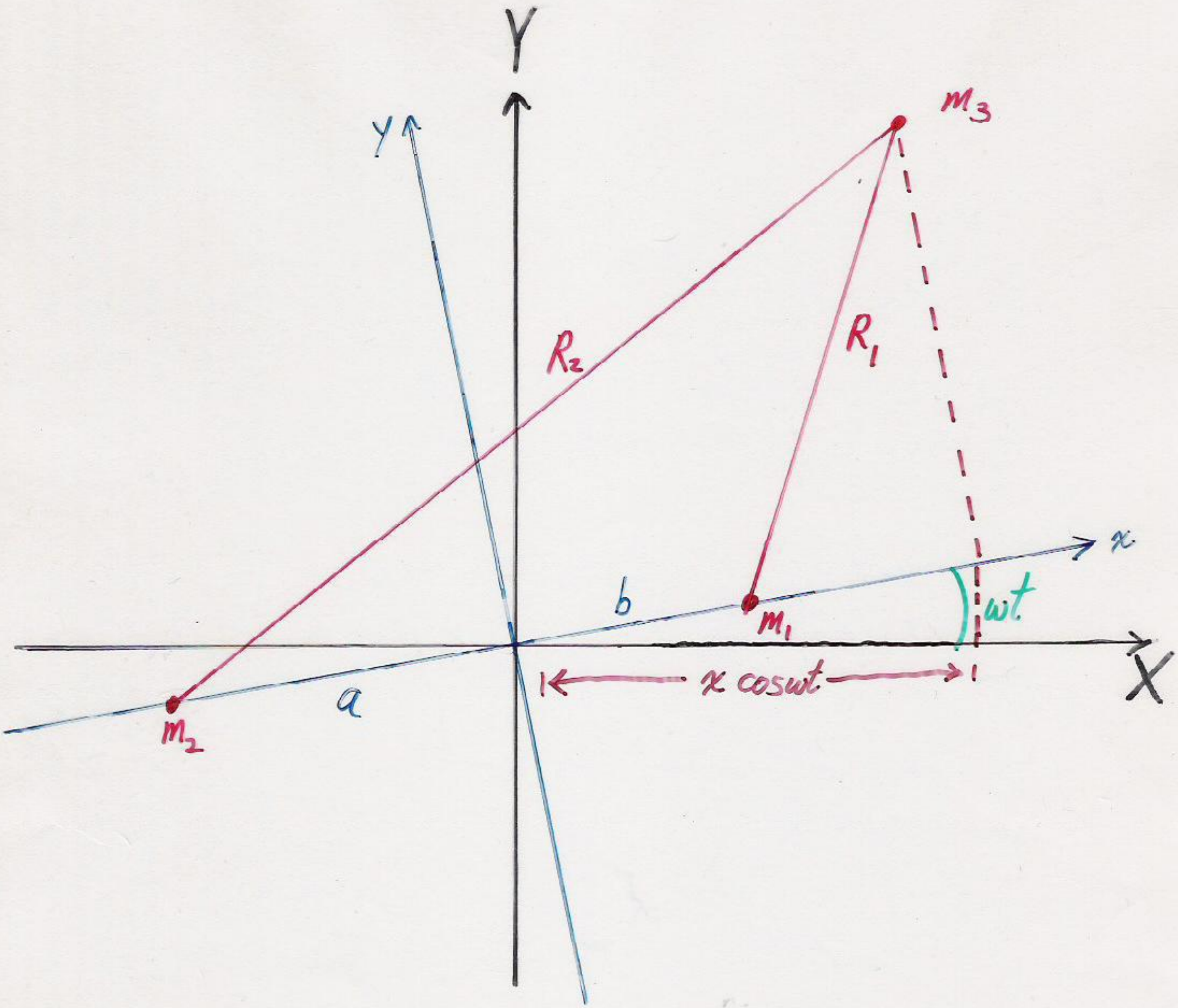
We will be primarily concerned with the motion of the very low mass particle m_3. Assume that particles m_1, m_2 are on elliptical orbits about the CM, like a 2-body system.

Define the position of m_3 in an inertial frame as:
$R = (X(t), Y(t))$.

Also define -
R_1 = the distance between m_1 & m_3
$= [(X-X_1)^2 + (Y-Y_1)^2]^{1/2}$.
R_2 = the distance between m_2 & m_3
$= [(X-X_2)^2 + (Y-Y_2)^2]^{1/2}$.

Choose the origin and coordinate direction such that m_1, m_2 are on the X-axis at $t = 0$, then,

$$X_1 = b\cos(\omega t), \qquad Y_1 = b\sin(\omega t),$$
$$X_2 = -a\cos(\omega t), \qquad Y_2 = -a\sin(\omega t).$$



$m_1 > m_2 >> m_3$

The gravitational force on m_3 equals $-Gm_1R_1/R_1^3 - Gm_2R_2/R_2^3$, and the equations of motion in the inertial frame are,

$$\ddot{X} = \frac{-Gm_1}{R_1^3}(X - b\cos(\omega t)) \quad - \quad \frac{Gm_2}{R_2^3}(X + a\cos(\omega t))$$

$$\ddot{Y} = \frac{-Gm_1}{R_1^3}(Y - b\sin(\omega t)) \quad - \quad \frac{Gm_2}{R_2^3}(Y + a\sin(\omega t))$$

Key to this problem: Transform to a rotating coordinate system. Jump on the carousel!

Specifically, define a "co-moving" radius r, by,

$$\vec{R} = R(\omega t)\vec{r},$$

then the m_1-m_2 axis (i.e., the x-axis) rotates with the two massive bodies.

More precisely, the transformation is given by,

$$\vec{X} = \vec{x}(t)\cos(\omega t) + \vec{y}(t)\sin(\omega t)$$

$$\vec{Y} = \vec{y}(t)\cos(\omega t) - \vec{x}(t)\sin(\omega t),$$

skip

~~Define $\vec{r}$ by $\vec{R} = R(\omega t)\vec{r}$.~~

~~The m_1-m_2 axis (i.e. x-axis) is fixed in the rotating frame.~~

~~More precisely, transformation given by,~~

position of m_3 in inertial system

$$\cancel{\vec{X} = \vec{x}(t)\cos\omega t + \vec{y}(t)\sin\omega t}$$

$$\cancel{\vec{Y} = \vec{y}(t)\cos\omega t - \vec{x}(t)\sin\omega t}$$

position of m_3 in moving coord. system

Velocity transform:

x-velocity of m_3 in moving system.

$$\dot{\vec{X}} = \dot{\vec{x}}\cos\omega t - \omega\vec{x}\sin\omega t + \dot{\vec{y}}\sin\omega t + \omega\vec{y}\cos\omega t$$

Acceleration: (along X-axis, in terms of x, y position)

$$\ddot{\vec{X}} = \ddot{\vec{x}}\cos\omega t - 2\omega\dot{\vec{x}}\sin\omega t - \omega^2\vec{x}\cos\omega t + \ddot{\vec{y}}\sin\omega t + 2\dot{\vec{y}}\omega\cos\omega t - \omega^2\vec{y}\sin\omega t$$

Take dot product with unit vector $\hat{x}$ along x-axis to get eq. of motion:

$$\ddot{\vec{X}}\cdot\hat{x} = \ddot{x}\cos\omega t - 2\omega\dot{y}\cos\omega t - \omega^2 x\cos\omega t$$

$$= \frac{-Gm_1}{R_1^3}(x\cos\omega t - b\cos\omega t) - \frac{Gm_2}{R_2^3}(x\cos\omega t + a\cos\omega t)$$

(Note: $\hat{x}\cdot\dot{\vec{x}} = \hat{x}\cdot\vec{y} = \hat{x}\cdot\ddot{\vec{y}} = 0$)

Similarly, for Y-component.

skip

Finally, the transformed equations of motion are:

$$\ddot{x} - 2\omega\dot{y} - \omega^2 x = \frac{-Gm_1}{R_1^3}(x-b) - \frac{Gm_2}{R_2^3}(x+a)$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y = -\left[\frac{Gm_1}{R_1^3} + \frac{Gm_2}{R_2^3}\right] y$$

↖ Coriolis term

Multiply the first eq. by $\dot{x}$, 2^{nd} by $\dot{y}$ and add,

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} - \omega^2(x\dot{x} + y\dot{y}) = \frac{-Gm_1}{R_1^3}\overbrace{\left[(x-b)\dot{x} + y\dot{y}\right]}^{\dot{(R_1^2)}}$$

$$- \frac{Gm_2}{R_2^3}\underbrace{\left[(x+a)\dot{x} + y\dot{y}\right]}_{\dot{(R_2^2)}}$$

(Note R_1, R_2 constant except for the motion of m_3.)

Include

Integrate (another energy integral), to get

$$\dot{x}^2 + \dot{y}^2 = 2F - C$$

with $F = \frac{\omega^2}{2}(x^2+y^2) + G\left(\frac{m_1}{R_1} + \frac{m_2}{R_2}\right)$

C = integration const.

Called Jacobi's Integral.

Though it is not the complete sol'n., we can learn a great deal from this integral, as in the 2-body prob.

Equilibrium Sol'ns. $\dot{x} = \ddot{x} = \dot{y} = \ddot{y} = 0$

Eqns. of motion give:

$$\omega^2 x - \frac{Gm_1}{R_1^3}(x-b) - \frac{Gm_2}{R_2^3}(x+a) = 0$$

$$\left[\omega^2 - \frac{Gm_1}{R_1^3} - \frac{Gm_2}{R_2^3}\right] y = 0$$

(Centripetal balances grav.)

Special case: equilibria on the line $y = 0$, top eq. becomes

$$\omega^2 x - \frac{Gm_1 \,\mathrm{sgn}(x-b)}{(x-b)^2} - \frac{Gm_2}{(x+a)^2} = 0$$

Third order eq. with 3 sol'ns.

If $y \neq 0$ the eqns. above have 5 sol'ns., called the Lagrange points L_{1-5}.

Now one can use Kepler's 3rd law to eliminate ω from the y-equation, and show that the non-collinear points L_4, L_5 have $R_1 = R_2 = a+b$. They form equilateral triangles with m_1 and m_2.

Lagrange Pt. Fig.

Stability Collinear points L_{1-3} are unstable. L4, L5 stable!

Zero Velocity Curves determine allowed regions for orbits with a given amount of energy.

K. Symon
1971

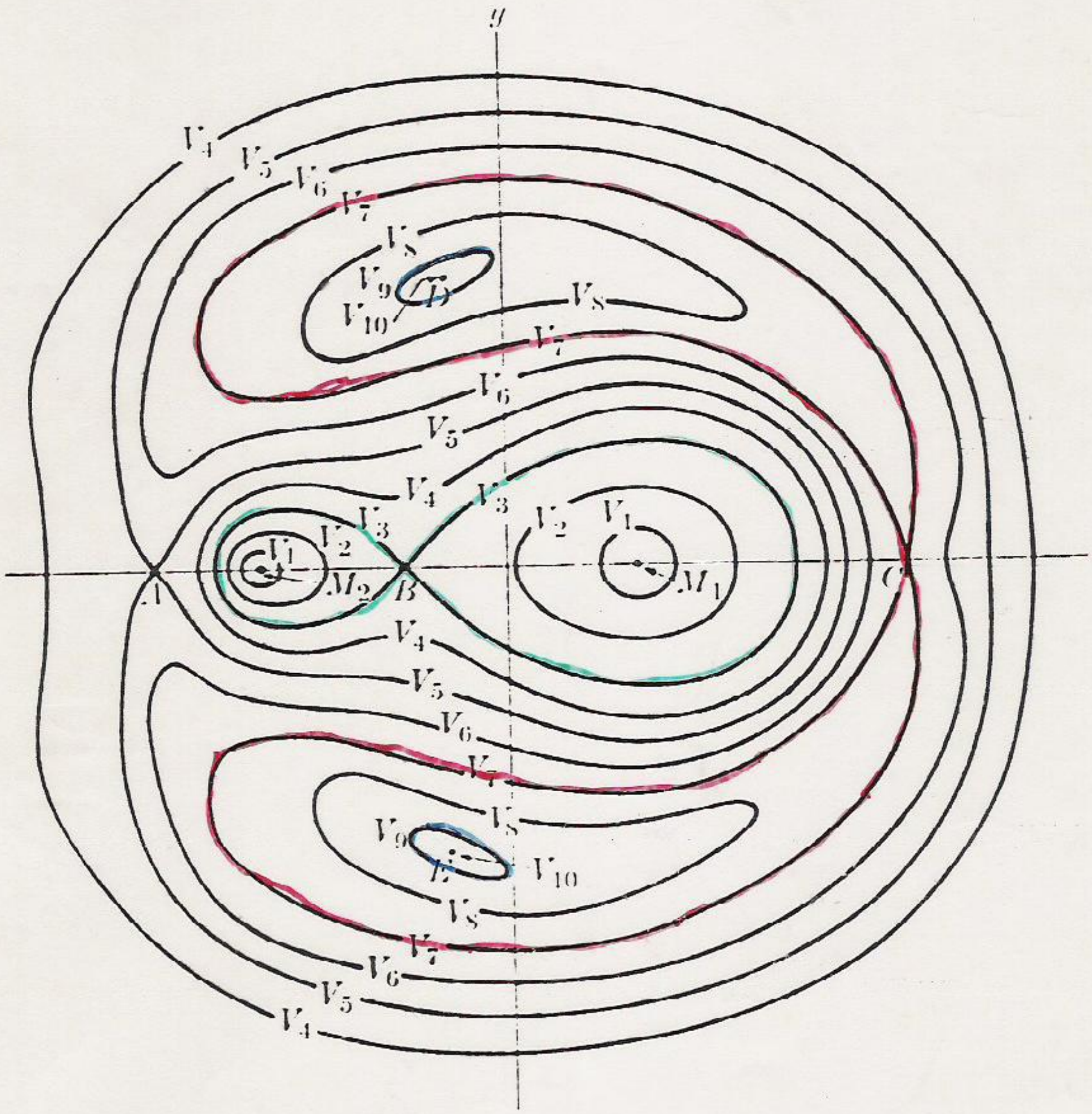


Fig. 7.8 Equipotential contours for 'V' (x, y).

Some applications of L.-point theory

① **Space colonies or factories**

Good, stable sites at L4,5, though solar perturbations are important.

② **"Trojan" asteroids**

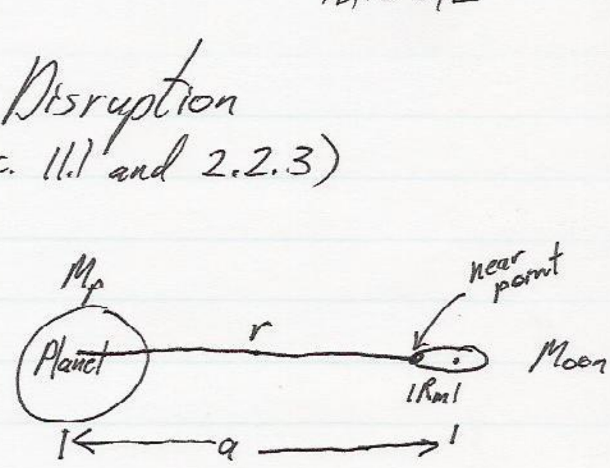
at Jupiter-Sun Lagrange pts.

③ **Roche-lobe overflow** in close binaries. Expanding envelope of giant star exceeds lobe boundary.

④ **Barred Galaxies**

somewhat like 3-body systems.

Hill Sphere and Roche Disruption
(de Pater and Lissauer sec. 11.1 and 2.2.3)



Centrifugal accel. on the moon:

$$\frac{V_\theta^2}{a} = \omega^2 a$$

For a circular orbit around the planet, $V_\theta^2 = \frac{GM_p}{a}$, so $\omega^2 = \frac{GM_p}{a^3}$

Since the near point and the center of the moon are attached to each other, the value of ω is the same for both. Therefore, the centrifugal accel. for the near point, at radius r, is –

$$\omega^2 r = \frac{GM_p}{a^3} r.$$

The total accel. at that point, called the 'effective gravity' is the difference between this and the grav. accel.

$$g_{eff} = GM_p \left(\frac{r}{a^3} - 1/r^2\right)$$

The differential tidal accel. between the near point and the center of the moon is,

$$R_m \frac{d g_{eff}}{dr} = GM_p R_m \left(\frac{1}{a^3} + \frac{2}{r^3}\right) = GM_p R_m \left(\frac{1}{a^3} + \frac{2}{(a-R_m)^3}\right)$$

$$\approx \frac{3 GM_p R_m}{a^3}$$

Note the different numerical factor than pure tidal accel. (because we've included centrifugal effects).

The moon's surface grav. $= \dfrac{-G M_m}{R_m^2}$

Balance between surface grav. and differential tidal accel. gives Roche (disruption) limit.

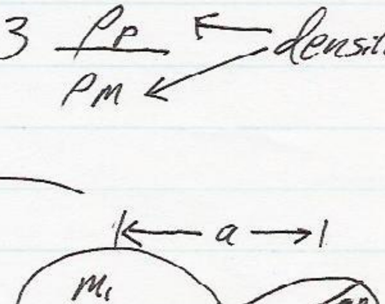
$$\frac{a^3}{R_m^3} = 3\,\frac{M_p}{M_m},$$

for given values of M_p, M_m, R_m, the value of 'a' given by this eq. is how close the moon can get before it is torn apart.

The masses can be hard to determine directly, so we can write:

$$\frac{a^3}{R_p^3} = 3\,\frac{M_p}{M_m}\,\frac{R^3}{R_p^3} = 3\,\frac{\rho_p}{\rho_m} \leftarrow \text{densities.}$$

The radius of the Hill sphere, in the restricted 3-body problem, follows from similar considerations.



$$\frac{R_H}{a} = \left(\frac{m_2}{3(m_1+m_2)}\right)^{1/3}$$