SDK

Newton & the "Last Word" on Orbits

Kepler: EMPIRICAL LAWS

- Based solely on a simple description
- Predictions enabled, - but -
- Not an explanation (what?, not why?)

NEWTON

- peculiar historical background
- precociousness
- Galileo influence, etc

→ Newton's Laws (Physics 221) address in full generality - motions of point masses
- subject to external forces

* - separability of orthogonal direction

1 - motion continues at constant velocity (speed + direction) unless a unbalanced external force acts (i.e. gravity, friction...)

2 - Change in velocity is proportional to force,
* in same direction as force, inversely with mass

$$\vec{F} = m\vec{a}$$

3 - Action = - Reaction $\vec{F}_1 = -\vec{F}_2$ $m_1\vec{a}_1 = -m_2\vec{a}_2$

Universal Gravitation

- Gravity is a universal, attractive force between bodies
- depends solely on mass and
- distance:

unit vector in radial dir.

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

inverse square law

m ←— r —→ M

note: $\vec{F} = m\vec{a}$, so $\vec{a} = -\hat{r}\frac{GM}{r^2}$ is grav. accel.

August 30
- HW Reminder
- reading Ch. 3

Measured value on Earth: $9.8\ m/s^2$

Hold this thought.

Newton on orbital motion

Calculus = instantaneous change

v_r v_t

- v_r is motion, radially, towards or away from ⊕
- $v_t \perp \hat{r}$, so no force effects

* from gravity.

∴ orbiting body (i.e. Moon) is in perpetual free-fall

Simplest case: circular motion — no net radial acceleration

constant centrifugal balance:

⑦

* Force balance = no net force = constant ENERGY } conservation of energy ; momentum

Inward force (gravity) = outward force (centrif. accel)

$$m\frac{GM}{r^2} = m\frac{V_\perp^2}{r}$$

$$V_\perp^2 = \frac{GM}{r} \;;\quad V = \sqrt{\frac{GM}{r}} \text{ is circular velocity}$$

Orbital period $= \frac{2\pi r}{V}$ — dep. on • orbit size • Mass of dominant object

So $\frac{2\pi r}{P} = \sqrt{\frac{GM}{r}}$

$$4\pi^2 r^2 = \frac{GM}{r} P^2 \longrightarrow \boxed{P^2/r^3 = \frac{4\pi^2}{GM}}$$

- Only works for moon if acceleration drops w/ r^2
- this is genesis of "inverse square law" for gravity

Look familiar?

Kepler #3! ($M = M_\odot$) ($= M_\odot + M_p$)

via Newton's Laws + U.G. can also derive Kepler #1 and Kepler #2 !

↳ universal laws + def. of mass, accel, etc shows underlying "cause" of Kepler's Laws

Kepler 1: Solutions to $\vec{F} = m\vec{a}$ in general produce

- orbits that are all conic sections,
- center of mass at one focus

⑧

Aero E 351 r_1

$$E_{tot} = \frac{1}{2}mv^2 - \frac{GMm}{r} \qquad \mu = \frac{M_1 M_2}{M_1 + M_2}$$

Kepler 1, cont'd — Conic Sections

- ellipse
  - bound orbit
  - neg. total energy
  - planets
  - e=0 (circular orbit) a special case
- parabola
  - 0 total energy
  - "just" unbound
  - comets
- hyperbola
  - + total energy
  - unbound
  - material ejected from s.s.

* Water fountain → jet shape? *

Kepler 2 ↳ "simply" conservation of angular momentum

$$\vec{L} = \vec{r} \times m\vec{v}, \quad \frac{d\vec{L}}{dt} = 0 \;\to\; mvr = mr\,r\frac{d\theta}{dt} = \text{const}$$

$$\text{Area} = \tfrac{1}{2}r^2 d\theta \;\to\; m\,2\frac{dA}{dt} = \text{const} \quad \boxed{dA/dt = \text{const}}$$

Escape Velocity? $E_{tot} \geq 0$ $\quad \frac{1}{2}mv^2 \geq \frac{GMm}{r}$

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} \quad \text{integrate from } r=\infty, v=0 \text{ to } r$$

$$\frac{1}{2}(\dot{r})^2\Big]_0^{v} = +\frac{GM}{r}\Big]_\infty^r$$

$$V^2 = \frac{2GM}{r}$$

$$V = \sqrt{2GM/r}$$

$$V^2 \geq \frac{2GM}{r} \qquad \boxed{V \geq \sqrt{\frac{2GM}{r}}}$$

"Point masses" vs. Macroscopic Bodies

(9)

Tidal Forces

- important for studying details of planetary / satellite interiors
- orbital evolution via transfer of orbital energy into deformations, rotational energy, and vice-versa

Differential Gravitational force

$$F_c = \frac{GMm}{R^2} \qquad F_A = \frac{GMm}{(R-r)^2} \qquad F_B = \frac{GMm}{(R+r)^2}$$

Force difference between A & C

$$F_A - F_c = GMm \cdot \left(\frac{1}{(R-r)^2} - \frac{1}{R^2}\right)$$

$$= \frac{GMm}{\cancel{R^2}} \left(\frac{R^2 - (R-r)^2}{R^2 (R^2 - 2rR + r^2)}\right)$$

$$= \frac{GMm}{R^2} \left(\frac{\cancel{R^2} - \cancel{R^2} + 2rR - r^2}{R^2 - 2rR + r^2}\right)$$

$$= \cancel{\frac{GMm}{R^3}\left(2r \ldots\right)}$$

Assume $r/R \ll 1$

$$= \frac{GMm}{R^2}\left[\frac{2R^2\left(\frac{r}{R}\right) - R^2\cancel{\left(\frac{r}{R}\right)^2}}{R^2\left(1 - 2\frac{r}{R} + \cancel{\left(\frac{r}{R}\right)^2}\right)}\right] \approx \frac{GMm}{R^2} \cdot 2\frac{r}{R}$$

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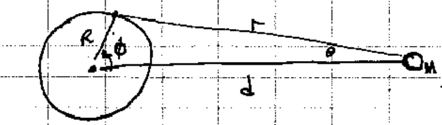
More generally



$$a_{core} = -\frac{GM}{d^2} \qquad a_{surf_{\parallel \text{ to } d}} = -\frac{GM}{r^2}\cos\theta$$

$$\cos\theta = \frac{d - R\cos\phi}{r} \Rightarrow a_{surf} = \frac{GM}{d^2}\left[\frac{d^3}{r^3} - \frac{d^2 R}{r^3}\cos\phi\right]$$

$$a_{surf} - a_{core} = -\frac{GM}{d^2}\left[\frac{d^3}{r^3} - \frac{d^2}{r^3}R\cos\phi - 1\right]$$

note $d = r\cos\theta + R\cos\phi$

if θ small, $\frac{d}{r} \approx 1 + \frac{R}{r}\cos\phi$

$$\frac{d^2}{r^2} \approx 1 + 2\frac{R}{r}\cos\phi + \cancel{\frac{R^2}{r^2}\cos^2\phi}$$

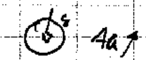
$$\frac{d^3}{r^3} = 1 + 3\frac{R}{r}\cos\phi + 3\left(\frac{R}{r}\right)^2\cos^2\phi + \left(\frac{R}{r}\right)^3\cos^3\phi$$

but θ small means $\frac{R}{r}$ small, so can go!

$$a_{surf} - a_{core} = -\frac{GM}{d^2}\left[\cancel{1} + 3\frac{R}{r}\cos\phi - \frac{1}{r}R\cos\phi\left[1 + 2\frac{R}{r}\cos\phi\right] - \cancel{1}\right]$$

$$\boxed{= -\frac{GM}{d^2}\left[2\frac{R\cos\phi}{r}\right] \approx -\frac{2GM}{d^3}R\cos\phi}$$

Compare tidal accel. to grav. accel (M)



$g = \frac{Gm}{r^2}$ $\qquad \Delta a = \frac{2GMr}{d^3}\cos\phi$

$\frac{\Delta a}{g} = \frac{2M}{m}\frac{r^3}{d^3}\cos\phi$ $\qquad \frac{\text{Moon tide}}{\text{earth } g} \approx 2.8\times10^{-8}$

Very small!

When is $\Delta a = g$?

$d = \left[\frac{2Mr^3}{m}\right]^{1/3}$ $\qquad$ but $m = \frac{4}{3}\pi r^3\rho_m$

$\qquad$ and $M = \frac{4}{3}\pi R^3\rho_M$

$d = \left[\frac{2M\,3}{4\pi\rho_m}\right]^{1/3} = \left[R^3 \times 2\left(\frac{\rho_M}{\rho_m}\right)\right]^{1/3}$

$d = 1.26\,R\,(\rho_M/\rho_m)$

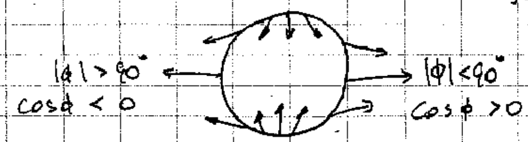
the "Roche Distance"

IORE GENERALLY

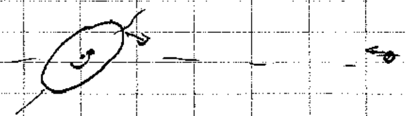
TIDES ⟶ DEFORMATION
(non-central effective potential)

- 1 day = 1 mo = 27 current days in distant future (when?)
- Moon & Earth much closer in past!

The "tidal bulge" — net effect of tidal force/accel



- Stretches out to a football shape, pointing at companion
- If rotating non-synchronously, ~~the~~ bulge pushes "ahead" of direct line of centers



↳ net torque pulls companion along
↳ add orbital energy at expense of rotational energy

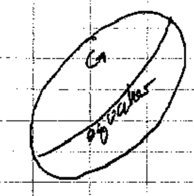
∴ ↳ orbit of (moon) moves out (longer period via Kepler 3)

~~rotation of (planet) slows~~

- currently, month lengthening ~ 0.014 s/century

∴ ↳ rotation of (planet) slows (moon already synchronous) $\oplus$ day lengthening by 0.0015 s/century

Another effect: - ⊕ rotation tilted w.r.t. Moon orbit.



- rotational flattening @ poles → ⊕

- Moon exerts a torque on equatorial bulge

↳ precession (like a gyro) of pole of ⊕ rotation

Wobble period ~ 26,000 years

Tidal Heating

- Small in ⊕-moon

- much bigger in resonant systems (ie. Jupiter's moons). More later